Intuitive Derivation of the Euler's Continued Fraction

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The Euler's continued fractions is usually proved by mathematical induction. The proof is presented in various sources. However, I couldn't find any derivation, or proof by other means. The proof by mathematical induction requires that one knows a-priory the general expression of what one's want to prove and, then, apply the method. However, how was the general expression obtained in the first pace? Where it comes from? Somebody must have developed the expression in order to have it proved by induction.

In this essay, I will try to figure out what was Euler's[3] intention when he conceived his continued fraction, and how he obtained it. I am not claiming that this was never done before; I have not thoroughly searched the literature. I am just trying to learn and convince myself.

I believe that in his ingenious and extensive work Euler had by some reason (somebody might know), came across with the following finite sum

$$a_0+a_0a_1+a_0a_1a_2+\ldots+a_0a_1a_2\cdots a_n$$

It is possible too that Euler wanted to obtain an expression free from long products and grouping symbols (namely, avoid the use of parentheses) or, just for fun[4], he decided to find a continued fraction for the expression above. Since computers were not available in his time, not even hand-held calculators, I believe that the latter was indeed his drive. Anyway, Euler's continued fractions is the following algebraic equation:

$$a_0+a_0a_1+\ldots+a_0a_1a_2\cdots a_n=\cfrac{a_0}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{1+a_2-\cfrac{\ddots}{\ddots\cfrac{a_{n-1}}{1+a_{n-1}-\cfrac{a_n}{1+a_n}}}}}} \tag{1}$$

The proof by mathematical induction starts by showing that it is valid for $n=1$, then assuming that it is valid for a given k and then, by a series of proper algebraic operations, to show that it is valid also for $k+1$. This is fine, and certainly proves that it is valid for any integer > 1. But it is not a proper derivation of Euler's continued fraction because one starts with the expression itself.

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2 ⓒ is the Copyleft symbol. If you don't know it, look here: https://en.wikipedia.org/wiki/Copyleft and here https://copyleft.org

3 Leonard Euler, 15 Apr 1707 – 18 Sep 1783 (https://en.wikipedia.org/wiki/Leonhard_Euler)

4 Many mathematical discoveries was just for fun

Let's then try to find a way to, at least, reach Euler's expression so that it can be proved by induction. Let's assume that Euler was searching for expressions without products and/or grouping symbols, and start with the simplest case:

$$a_0+a_0a_1$$

Now, we get rid of the product a_0a_1 by appropriate algebraic manipulations:

$$a_0+a_0a_1=a_0(1+a_1)=\frac{a_0}{\frac{1}{1+a_1}}=\frac{a_0}{\frac{1+a_1-a_1}{1+a_1}}=\frac{a_0}{1-\frac{a_1}{1+a_1}} \qquad (2)$$

Note that the final expression is free from multiplications (OK, one may say that division is the multiplication by the reciprocal, but let's keep rigor out of this). But the very second expression is also free of multiplication. Why not just stop there? Well, because the expression itself (without the equal sign properly located) is ambiguous. Consider this expression:

$$\frac{5}{\frac{3}{2}}$$

In the absence of a equal sign properly located, or by grouping it, one can not tell if this expression is equal to 3.333… or equal to 0.8333... By just defining a priority for the division will not resolve the problem either, because one might need exactly the expression that does not follow the priority, and must group it using, for example, parenthesis: If they are grouped, then their value are unquestionable:

$$\frac{5}{\left(\frac{3}{2}\right)} \qquad\qquad \frac{\left(\frac{5}{3}\right)}{2}$$

From each of these expressions, one can undoubtly find its value. By using the format of the rightmost expression in Eq. 2 the ambiguity disappears. Using the previous fractions, one can chose to use:

$$\frac{5}{1+\frac{1}{2}} \qquad\qquad \frac{1+\frac{2}{3}}{2}$$

Now, one can immediately tell the correct values of these expressions, without using operation's priorities or grouping them. The ambiguity vanishes.

Back to the discussion, the final expression for the sum of product $a_0+a_0a_1$ is

$$a_0+a_0a_1=\frac{a_0}{1-\frac{a_1}{1+a_1}}$$

By analogy, one could guess that the expression for $a_0+a_0a_1+a_0a_1a_2$ could extend naturally using the term $1-\frac{a_2}{1+a_2}$ for $1+a_1$ resulting in

$$a_0+a_0a_1+a_0a_1a_2=\frac{a_0}{1-\frac{a_1}{1-\frac{a_2}{1+a_2}}}\ (\textbf{\textit{wrong}})$$

If one just resolves the fractions on the right side the result will prove totally incongruent.

The correct way is to proceed and find an expression for $a_0+a_0a_1+a_0a_1a_2$ using a similar approach as above. We start with:

$$a_0+a_0a_1+a_0a_1a_2=\frac{a_o}{\dfrac{1}{1+a_1+a_1a_2}}$$

To avoid the above discussed ambiguity, we continue manipulating the denominator fraction:

$$\cdots=\frac{a_o}{\dfrac{1}{1+a_1+a_1a_2}}=\frac{a_o}{\dfrac{1+(a_1+a_1a_2)-(a_1+a_1a_2)}{1+a_1+a_1a_2}}=\frac{a_o}{1-\dfrac{a_1+a_1a_2}{1+a_1+a_1a_2}}$$

Note that the last term at the right is not ambiguous but still uses products (twice). Therefore, we keep going:

$$\cdots=\frac{a_o}{1-\dfrac{a_1+a_1a_2}{1+a_1+a_1a_2}}=\frac{a_o}{1-\dfrac{a_1(1+a_2)}{1+a_1+a_1a_2}}=\frac{a_o}{1-\dfrac{a_1}{\dfrac{1+a_1+a_1a_2}{1+a_2}}}$$

Now, we must get rig of the remaining a_1a_2 in the rightmost expression:

$$\cdots=\frac{a_o}{1-\dfrac{a_1}{\dfrac{1+a_1+a_1a_2}{1+a_2}}}=\frac{a_o}{1-\dfrac{a_1}{\dfrac{1+a_1+a_1a_2+a_2-a_2}{1+a_2}}}=\frac{a_o}{1-\dfrac{a_1}{\dfrac{(1+a_1)(1+a_2)-a_2}{1+a_2}}}$$

Simplifying the lower rightmost fraction gives:

$$a_0+a_0a_1+a_0a_1a_2=\frac{a_o}{1-\dfrac{a_1}{1+a_1-\dfrac{a_2}{1+a_2}}} \tag{3}$$

Let's compare this expression with the Eq. 2 and see what we can conclude when $a_0a_1a_2$ was added:

$$a_0+a_0a_1=\frac{a_0}{1-\dfrac{a_1}{1+a_1}} \qquad\qquad a_0+a_0a_1+a_0a_1a_2=\frac{a_o}{1-\dfrac{a_1}{1+a_1-\dfrac{a_2}{1+a_2}}}$$

At first view, we can say that advancing from $a_0+a_0a_1$ to $a_0+a_0a_1+a_0a_1a_2$ the term $1+a_1$ was replaced by $1+a_1-\dfrac{a_2}{1+a_2}$. One can (correctly) guess that advancing to $a_0+a_0a_1+a_0a_1a_2+a_0a_1a_2a_3$ all we need to do is replace $1+a_2-\dfrac{a_3}{a+a_3}$ for $1+a_2$. On the other hand, someone could say that a natural (wrong) step would be using $1+a_1+a_2-\dfrac{a_3}{a+a_3}$ for $1+a_2$ by comparing the terms in red in the following expression:

$$a_0+a_0a_1+a_0a_1a_2+a_0a_1a_2a_3=\cfrac{a_o}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{1+a_1+a_2-\cfrac{a_3}{1+a_3}}}}\ \text{(wrong)}$$

Although we could simply demonstrate that the correct expression for $a_0+a_0a_1+a_0a_1a_2+a_0a_1a_2a_3$ is as guessed, we will now find the expression for the sum of products above by direct calculation as we've done for the previous cases, and see if we can obtain a stable pattern[5]. As usual, we start with the equation

$$a_0+a_0a_1+a_0a_1a_2+a_0a_1a_2a_3=\frac{a_1}{\cfrac{1}{1+a_1+a_1a_2+a_1a_2a_3}}=\frac{a_1}{\cfrac{1+(a_1+a_1a_2+a_1a_2a_3)-(a_1+a_1a_2+a_1a_2a_3)}{1+a_1+a_1a_2+a_1a_2a_3}}$$

$$\cdots=\frac{a_1}{\cfrac{1+a_1+a_1a_2+a_1a_2a_3-(a_1+a_1a_2+a_1a_2a_3)}{1+a_1+a_1a_2+a_1a_2a_3}}=\frac{a_1}{1-\cfrac{a_1+a_1a_2+a_1a_2a_3}{1+a_1+a_1a_2+a_1a_2a_3}}$$

$$\cdots=\frac{a_1}{1-\cfrac{a_1+a_1a_2+a_1a_2a_3}{1+a_1+a_1a_2+a_1a_2a_3}}=\frac{a_1}{1-\cfrac{a_1(1+a_2+a_2a_3)}{1+a_1+a_1a_2+a_1a_2a_3}}$$

$$\cdots=\frac{a_1}{1-\cfrac{a_1(1+a_2+a_2a_3)}{1+a_1+a_1a_2+a_1a_2a_3}}=\frac{a_1}{1-\cfrac{a_1}{\cfrac{1+a_1+a_1a_2+a_1a_2a_3}{1+a_2+a_2a_3}}}$$

Now we factor out the term $1+a_1$ by adding and subtracting $a_2+a_2a_3$ to the numerator of the further lower fraction to the right.

$$\cdots=\frac{a_1}{1-\cfrac{a_1}{\cfrac{1+a_1+a_1a_2+a_1a_2a_3}{1+a_2+a_2a_3}}}=\frac{a_1}{1-\cfrac{a_1}{\cfrac{1+a_1+a_1a_2+a_1a_2a_3+(a_2+a_2a_3)-(a_2+a_2a_3)}{1+a_2+a_2a_3}}}$$

And since $1+a_1+a_1a_2+a_1a_2a_3+(+a_2+a_2a_3)=(1+a_1)(1+a_2+a_2a_3)$, we obtain:

$$\cdots=\frac{a_1}{1-\cfrac{a_1}{\cfrac{(1+a_1)(1+a_2+a_2a_3)-(a_2+a_2a_3)}{1+a_2+a_2a_3}}}=\frac{a_1}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2+a_2a_3}{1+a_2+a_2a_3}}}$$

$$\cdots=\frac{a_1}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2+a_2a_3}{1+a_2+a_2a_3}}}=\frac{a_1}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2(1+a_3)}{1+a_2+a_2a_3}}}$$

5 A *stable pattern* is not risk free. The famous example is the sequence of the five first prime numbers, 1, 2, 3, 5, 7 and one could say that, despite the anomalous 2, "well, we reached a stable pattern and it seems that all odd numbers are primes".

$$\cdots=\cfrac{a_1}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2(1+a_3)}{1+a_2+a_2a_3}}}=\cfrac{a_1}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{\cfrac{1+a_2+a_2a_3}{1+a_3}}}}$$

And then, adding and subtracting a_3 to the numerator of the further lower fraction to the right and factoring out $1+a_2$ we get:

$$\cdots=\cfrac{a_1}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{\cfrac{1+a_2+a_2a_3+a_3-a_3}{1+a_3}}}}=\cfrac{a_1}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{\cfrac{(1+a_2)(1+a_3)-a_3}{1+a_3}}}}$$

$$a_0+a_0a_1+a_0a_1a_2+a_0a_1a_2a_3=\cfrac{a_1}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{1+a_2-\cfrac{a_3}{1+a_3}}}} \tag{4}$$

After these third attempt of the process, we can feel some confidence about the general form of the sum of products. The pattern inspires a next attempt by adding the term $\frac{-a_4}{1+a_4}$ after the term $1+a_3$ and so on for a general expression, which one can try to prove by induction. Evidently, the general expression is exactly the one presented in the beginning of the article (Eq. 1), repeated here for comparison:

$$a_0+a_0a_1+a_0a_1a_1+\ldots+a_0a_1a_2\cdots a_n=\cfrac{a_0}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{1+a_2-\cfrac{\ddots}{\ddots\cfrac{a_{n-1}}{1+a_{n-1}-\cfrac{a_n}{1+a_n}}}}}}$$

Comparing the terms, we see the pattern. From here, the reader can use this expression in a mathematical induction, which exists in many sources.[6]

6 For example, https://en.wikipedia.org/wiki/Euler's_continued_fraction_formula